WESTFIELDS SPORTS

YEAR 12 TRIAL EXAM

2008

MATHEMATICS

Reading time – 5 minutes
Working Time - 3 hours

INSTRUCTIONS TO STUDENTS:

- check that you have the correct paper
- approved calculators may be used
- use a new page for the start of each question
- all necessary working must be shown
- start each question on a new page

Total marks - 120

Attempt Questions 1 - 10
All questions are of equal value

Question 1

Marks

a) Evaluate, correct to three significant figures, 2

$$\sqrt{\frac{(3.024)^3}{25.5 - 13.018}}$$

b) Solve for x: $x^3 = 4x^2$ 2

c) Find the primitive of : $\frac{1}{3e^x}$ 1

d) Simplify : $\frac{1}{m^2 - 4m + 3} - \frac{1}{m^2 - 1}$ 3

e) Solve the pair of simultaneous equations 2

$$x - 2y = 1$$

$$xy = 1$$

f) Find the integers a and b such that $\frac{1}{2-\sqrt{3}} = a + b\sqrt{3}$ 2

Question 2

a) Write down the derivatives of :

(i) $(3x+4)^7$ 2

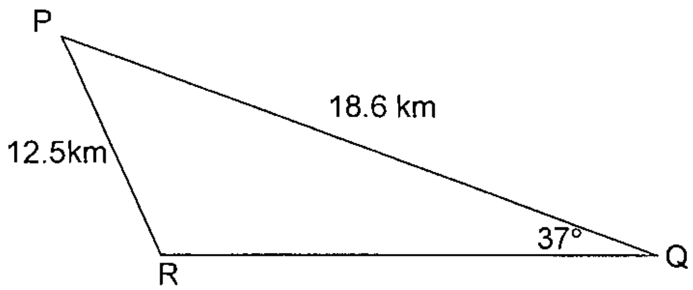
(ii) $x^3 e^x$ 2

(iii) $\frac{3x}{\sin x}$ 2

b) Find the exact value of $\tan\frac{\pi}{3} + \operatorname{cosec}\frac{\pi}{4}$ 2

c) Consider the quadratic function $x^2 - (k+2)x + 4 = 0$ 2
For what value of k does the quadratic function have real roots?.

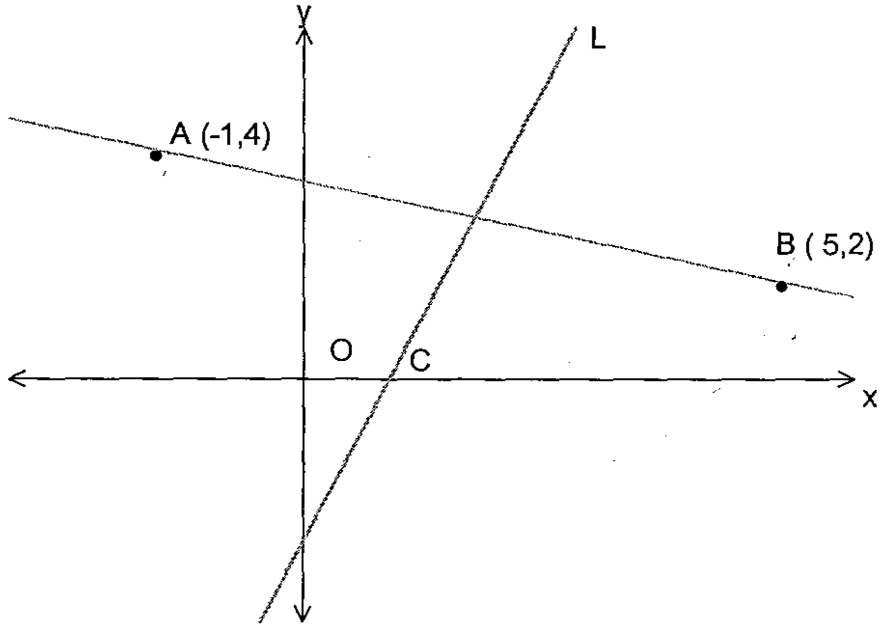
d) 2



In the diagram above, PQ = 18.6 km, PR = 12.5km and ∠PQR =37°. ∠PRQ is obtuse. Find the size of ∠PRQ correct to the nearest minute.

Question 3 **Marks**

a)



The diagram above shows the points A(-1,4) and B(5,2). The line L has equation $3x - y - 3 = 0$ and cuts the x-axis at C.

(i) Show that the length of AB is $2\sqrt{10}$ units. 1

(ii) Find the coordinates of M, the midpoint of AB. 1

(iii) Find the gradient of AB. 1

(iv) Show that the equation of AB is $x + 3y - 11 = 0$ 1

(v) Prove that L is perpendicular bisector of AB. 2

(vi) Find the coordinates of C. 1

(vii) Write down the equation of the circle with AB as a diameter. 1

b) α and β are the roots of the equation $x^2 - 6x + 10 = 0$. Find the values of:

(i) $\alpha + \beta$ 1

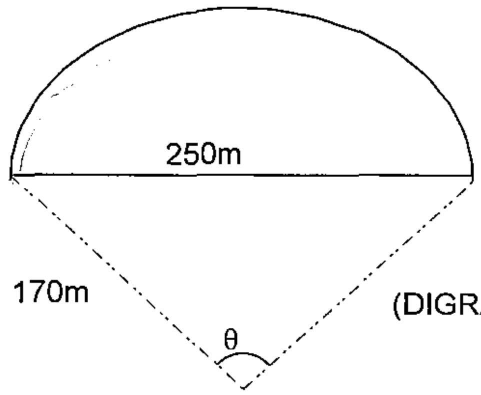
(ii) $\alpha\beta$ 1

(iii) $(\alpha + 1)(\beta + 1)$ 2

Question 4

Marks

a) A straight road was constructed to cut a dangerous bend on a country road. It was found that the bend was part of an arc of radius 170 metres and the straight road was 250 metres long.



(i) Use the cosine rule to find the size of θ correct to the nearest degree. 2

(ii) Find the distance by which the old road was shortened. Answer correct to the nearest metre. 3

b) For the parabola $16y = x^2$, write down the:

(i) coordinates of the focus 2

(ii) equation of the directrix. 1

c) (i) Sketch the graph of $y = -2\cos x$ for $0 \leq x \leq 2\pi$ 2

(ii) On the same axes, sketch the graph of $y = -2\cos x - 1$ for $0 \leq x \leq 2\pi$ 2

Question 5 **Marks**

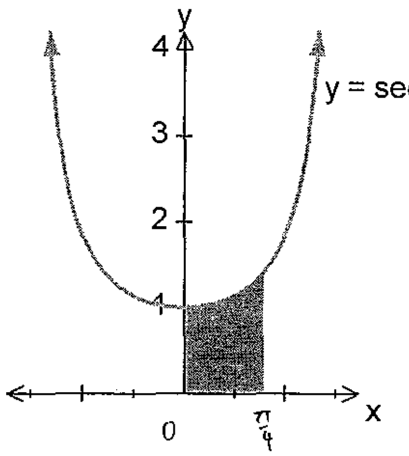
a)Find the equation of the

(i) tangent and 4

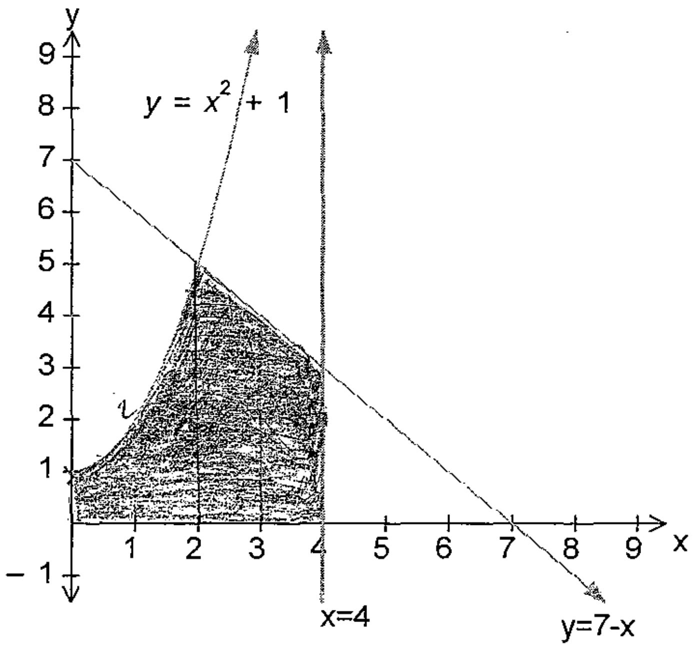
(ii) the normal to the curve $y = x\sin x$ at the point $(\frac{\pi}{2}, \frac{\pi}{2})$ 2

b) The shaded region which lies between the x axis and the curve y = secx 3

from x = 0 to $x = \frac{\pi}{4}$ is rotated about the x axis to form a solid. Find the volume of the solid.



c)



Use Simpson's Rule with five function values to find an approximation of the Shaded area. 3

Question 6 **Marks**

a) Given that $\sin\theta = \frac{3}{4}$ and $0^{\circ} < \theta < 90^{\circ}$, find as a single expression with rational denominator, the exact value of:

(i) $\cos\theta$ 2

(ii) $\cos\theta + \tan\theta$ 2

b) Find all values of θ such that $\sin 2\theta = 1$ and $0 \leq \theta \leq 2\pi$ 2

c) Solve the inequality $4x - x^2 > 0$ 2

d) For what value of m does the line $y = m(x+1)$ have no intersection with the parabola $y=2x^2$? 2

e) Solve the equation $e^x - 9e^{-x} = 0$ 2

Question 7.

a) The function f(x) is defined by the rule $f(x) = \begin{Bmatrix} 0 \text{ if } x \leq 0, \\ 2x \text{ if } x > 0 \end{Bmatrix}$

(i) Sketch the function $f(x)$, from $x = -2$ to $x = 2$ 2

(ii) Evaluate $\int_{-2}^{2} f(x)\, dx$ 1

b) The function f(x) is defined by the rule $f(x) = 9x(x-2)^2$ in the domain $-1 \leq x \leq 3$.

(i) find the x and y intercepts 2

(ii) find the stationary points and determine their nature. 3

(iii) find the values of the end-points 2

(iv) draw a sketch of the graph of $y = f(x)$, showing clearly the turning points, the intercepts and the end-points. 2

Question 8 **Marks**

a) Evaluate the following integrals: 6

(i) $\int_1^2 \frac{1}{x^3} dx$.

(ii) $\int_1^4 e^{3x} dx$

(iii) $\int_0^{\frac{\pi}{8}} \sec^2 2x \; dx$

b) Find $\int \frac{x}{x^2+4} dx$ 2

c) The population P of a town is growing at a rate proportional to the town's current population. The population at time t years is given by $P = Ae^{kt}$, where A and k are constants.

The population 20 years ago was 100 000 people and today the population of the town is 150 000 people.

(i) Find the value of A 1

(ii) Find the value of k 1

(iii) Find the population that will be present 20 years from now. 2

Question 9

a) Find $\frac{dy}{dx}$ given that $y = \log_e\left(\frac{2x+1}{3x-7}\right)$ 2

b) A particle moves in a straight line so that it's velocity, v metres per second, at time t is given by $v = 3 - \frac{2}{1+t}$.

The particle is initially 1 metre to the right of the origin.

(i) Find an expression for the position x, of the particle at time t. 2

(ii) Explain why the velocity of the particle is never 3 metres per second 1

(iii) Find the acceleration of the particle when t = 2 seconds. 2

c) (i) Show that $(\text{cosec}^2 A - 1)\sin^2 A = \cos^2 A$. 2

(ii) Hence, or otherwise solve $(\text{cosec}^2 A - 1)\sin^2 A = \frac{3}{4}$ for $-\pi \leq A \leq \pi$ 3

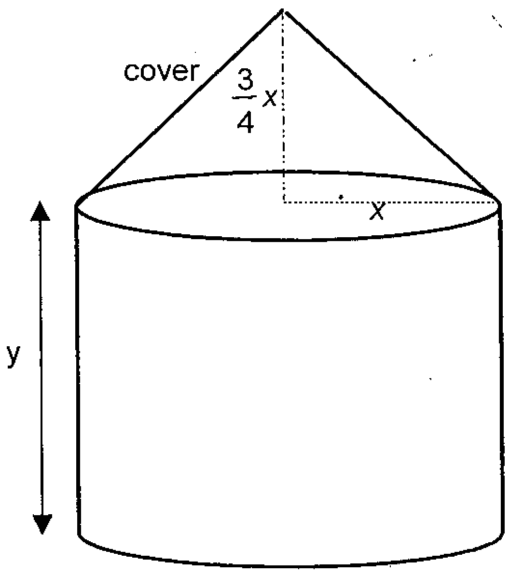
Question 10 **Marks**

An open cylindrical container is to hold 16m^3 of grain.

a) If the radius of the container is x metres and it's height is y metres, show that $y = \dfrac{16}{\pi x^2}$. 2

Hence show that the surface area, in square metres, of the container (sides and base) is $\pi x^2 + \dfrac{32}{x}$. 3

b) A conical cover of height $\dfrac{3}{4}x$ metres is placed on top to form a silo. Given that the surface area of an open cone of radius r, and slant height s is $\pi r s$, show that the surface area in square metres, of this cover is $\dfrac{5\pi x^2}{4}$. 3



c) The cost per unit area of the cover is 50% more than the cost per unit area of the sides and base. If k dollars per square metre is the cost per unit area of the sides and base, show that the total cost C in dollars of the silo (cover, sides and base) is given by

$C = k(\dfrac{23}{8}\pi x^2 + \dfrac{32}{x})$ 2

d) Find the value of x which minimises the total cost. 2

QUESTION 1

a) $1.488437459 = 1.49$ ✓

b) $x^3 = 4x^2$

$x^3 - 4x^2 = 0$

$x^2(x-4) = 0$

$\therefore x = 0, \quad x = 4$ ✓

c) $\frac{1}{3}\cancel{x}e^{-x} \Rightarrow -\frac{1}{3}e^{-x} = \frac{-1}{3e^x}$ ✓

d) $\frac{1}{m^2-4m+3} - \frac{1}{m^2-1}$

$\frac{1}{(m-1)(m-3)} - \frac{1}{(m+1)(m-1)}$

$\frac{m+1 - m+3}{(m-1)(m-3)(m+1)}$

$= \frac{4}{(m-1)(m+1)(m+3)}$ ✓

e) $\frac{1}{Y} - 2Y = 1 \Rightarrow 1 - 2Y^2 = Y$

$0 = 2Y^2 + Y - 1$

$(2Y-1)(Y+1) = 0$

$Y = \frac{1}{2}, \quad Y = -1$

$x = 2 \quad x = -1$ ✓

f) $\frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}$

$\therefore a = 2, \quad b = 1$ ✓

2

$(3x+4)^6 \cdot 3 = 21(3x+4)^6$

$x^3 e^x \Rightarrow 3x^2 e^x + x^3 e^x = x^2 e^x (3+x)$

$\frac{3x}{\sin x} \Rightarrow \frac{3\sin x - 3x\cos x}{\sin^2 x}$

b) $\tan\frac{\pi}{3} + \operatorname{cosec}\pi/4$

$= \tan 60 + \operatorname{cosec} 45$

$= \sqrt{3} + \frac{\sqrt{2}}{1}$

$= \sqrt{3} + \sqrt{2}$

c) $x^2 - (K+2)x + 4 = 0$

$\Delta \geq 0$

$(K+2)^2 - 4(1)(4) \geq 0$

$(K+2)^2 \geq 16$

$K+2 \geq \pm 4$

$\therefore K \geq 4, \quad K \leq -4$

$-6 \geq K \geq 2$

d) $\frac{\sin R}{18.6} = \frac{\sin 37}{12.5}$

$\sin R = \frac{18.6 \times \sin 37}{12.5}$

$\angle R = 116^\circ 26'$

QUESTION #3

(i) $AB = \sqrt{(5+1)^2 + (2-4)^2}$

$= \sqrt{36+4}$

$= \sqrt{40}$

$= 2\sqrt{10}$ ✓

(ii) $M = \left(\frac{5-1}{2}, \frac{4+2}{2}\right) = (2,3)$ ✓

(iii) $m = \frac{2}{-6} = -\frac{1}{3}$ ✓

(iv) $Y - 4 = -\frac{1}{3}(x+1)$

$3Y - 12 = -x - 1$

$3Y = -x + 11 \Rightarrow x + 3Y - 11 = 0$. ✓

(v) Gradient of $L = +3$ $\therefore M_{AB} \times M_L = 3 \times -\frac{1}{3} = -1$

and $(2,3)$ lies on L $[(3(2) - 3 - 3 = 0]$

(vi) x-intercept, $Y = 0$

~~$3(0) - Y - 3 = 0$~~ $3x - 0 - 3 = 0$

~~$-3 =$~~ $3x = 3$

$x = 1$

$\therefore C$ is $(1,0)$ ✓

(V) Equation of circle:

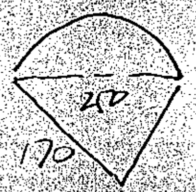
$(x-2)^2 + (y-3)^2 = 10$ ✓

b) (i) $\alpha + \beta = 6$ ✓

(ii) $\alpha\beta = 10$ ✓

(iii) $(\alpha+1)(\beta+1) = \alpha\beta + \alpha + \beta + 1 = 6 + 10 + 1 = 17$

QUESTION 4



(i) $\cos\theta = \dfrac{170^2 + 170^2 - 250^2}{2(170)(170)} = -0.081314878$

$\theta = 95^\circ$ ✓

(ii) Arc length $= \dfrac{95}{360} \times 2 \times \pi \times 170 = 282$ km.

$\therefore$ The distance was shortened by $282 - 250$

$= 32$ km. ✓

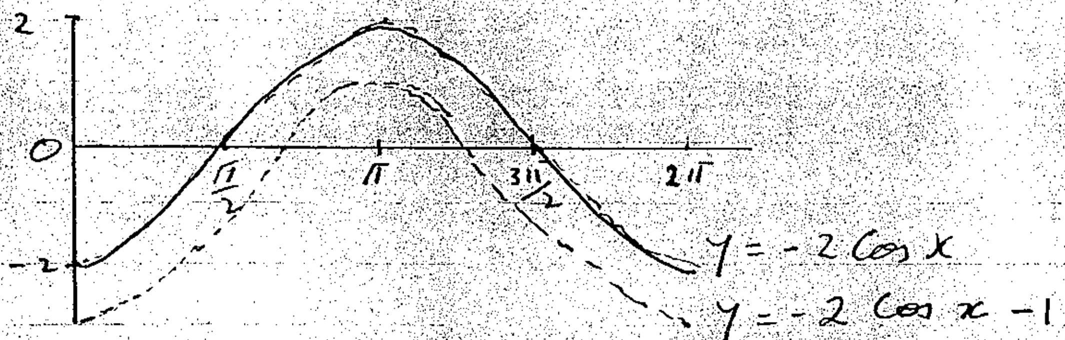
b) $16y = x^2$ $\qquad 4a = 16$

$a = 4$

(i) $(0, 4)$ ✓

(ii) $y = -4$ ✓

c) (i) $y = -2\cos x$



✓

QUESTION 5

a)(i) $y = x\sin x$ $\qquad \frac{dy}{dx} = \sin x + x\cos x$

At $\frac{\pi}{2}$, $\frac{dy}{dx} = 1 + \frac{\pi}{2}.0 = 1$

Equation of tangent $y - \frac{\pi}{2} = 1(x - \frac{\pi}{2})$

$y = x$

(ii)

Equation of Normal:

$y - \frac{\pi}{2} = -1(x - \pi/2)$

$y = -x + \pi$

b) $V = \pi\int_0^{\pi/4} (\sec x)^2\, dx$

$= \pi\left[\tan x\right]_0^{\pi/4}$

$= \pi\left[\tan \pi/4 - \tan 0\right]$

$= \pi$ unit^3.

c) $A \doteqdot \frac{1}{3}\left[0 + 3 + 2(5) + 4(2+4)\right]$

0	1	2	3	4
0	2	5	4	3
y_0	y_1	y_2	y_3	y_4

$= \frac{1}{3}\left[3 + 10 + 24\right]$

$= \frac{37}{3}$

$\doteqdot 12\frac{1}{3}$

QUESTION 6

a) (i) $\cos\theta = \frac{\sqrt{7}}{4}$ ✓

(ii) $\cos\theta + \tan\theta = \frac{\sqrt{7}}{4} + \frac{3}{\sqrt{7}}$ $= \frac{\sqrt{7}}{4} + \frac{3\sqrt{7}}{7}$

$= \frac{7\sqrt{7} + 12\sqrt{7}}{28}$

$= \frac{19\sqrt{7}}{28}$ ✓

b) $\sin 2\theta = 1$

$2\theta = \frac{\pi}{2}, \frac{5\pi}{2}$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ ✓

c) $x^2 - 4x > 0$

$x(x-4) \cancel{=} 0$

$x \cancel{=} 0, 4$

$0 \le x \le 4$

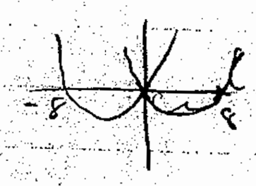
d) $y = m(x+1)$ $y = 2x^2$ $\therefore 2x^2 = mx + m$

$= mx + m$

$2x^2 - mx \cancel{+} m = 0$

$\therefore$ No intersection, $b^2 - 4ac < 0$

$m^2 - 4(2)(-m) < 0$



$m^2 + 8m < 0$

$m(m+8) < 0$

$m = 0, m = -8$ $-8 < m < 0$ ✓

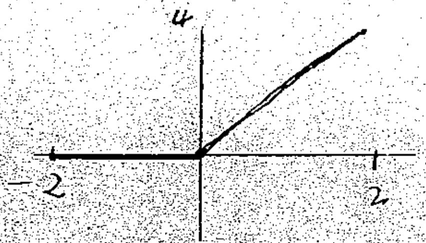
$-8 \cancel{\le} m < 0$

e) $e^x - 9e^{-x} = 0$ $e^x - \frac{9}{e^x} = 0$ $e^{2x} = 9$

$\frac{e^{2x} - 9}{e^{x}} = 0$ $x = \frac{\ln 9}{2}$

QUESTION 7

a) (i)



(ii) $\int_{-2}^{2} f(x)\,dx = \frac{1}{2} \times 2 \times 4 = 4 \text{ unit}^2$

b) $f(x) = 9x(x-2)^2$

(i) y-int, $x = 0 \quad \therefore 9(0)(0-2)^2 = 0$

x-int, $y = 0 \quad \therefore 9x(x-2)^2 = 0 \quad \therefore x = 0, 2$

(ii) $f'(x) = 9(x-2)^2 + 9x[2(x-2)]$

$= 9(x-2)^2 + 9x(2x-4)$

$= 9(x^2-4x+4) + 18x^2 - 36x$

$= 9x^2 - 36x + 36 + 18x^2 - 36x$

$= 27x^2 - 72x + 36$

$9x(x^2-4x+4)$

$9x^3 - 36x^2 + 36x$

$27x^2 - 72x + 36$

$f'(x) = 0 \quad \therefore 27x^2 - 72x + 36 = 0 \quad \div 9$

$3x^2 - 8x + 4 = 0$

$(3x-2)(x-2) = 0$

$x = 2/3, \quad x = 2$

$f(2/3) = 9(\frac{2}{3})(x-\frac{3}{2})^2$

$= \frac{6}{2} \cdot \frac{32}{9} = (\frac{32}{3}, 0)$

$9(2)(2-2)^2 = 0 \quad (2,0)$

Nature: $f''(x) = 54x - 72$

$f''(\frac{2}{3}) = 54(\frac{2}{3}) - 72 < 0$

(MAX

$f''(2) = 54(2) - 72 > 0$ MI

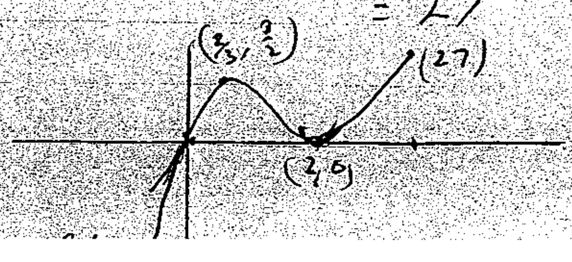
$f(-1) = 9(-1)(-1-2)^2$

$= -9(9)$

$= -81$

$f(3) = 9(3)(3-2)^2$

$= 27$



QUESTION 8

a) (i) $\displaystyle\int_1^2 x^{-3}\,dx = \left[\frac{-1}{2x^2}\right]_1^2 = \left[\frac{-1}{8} + \frac{1}{2}\right] = \frac{3}{8}$ ✓

(ii) $\displaystyle\int_1^4 e^{3x}\,dx = \left[\frac{1}{3}e^{3x}\right]_1^4 = \left[\frac{1}{3}e^{12} - \frac{1}{3}e^{3}\right]$

$= \frac{1}{3}e^3\left(e^9 - 1\right)$ ✓

(iii) $\displaystyle\int_0^{\pi/8} \sec^2 2x\,dx = \left[\frac{1}{2}\tan 2x\right]_0^{\pi/8} = \frac{1}{2}$ ✓

b) $\displaystyle\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln(x^2+4) + C$ ✓

c) i) $A = 100\,000$ ✓

(ii) $150\,000 = 100\,000\,e^{k(20)}$

$1.5 = e^{20k}$

$\frac{\ln 1.5}{20} = k$ ✓

(iii) $P = 100\,000\,e^{40 \times \frac{\ln 1.5}{20}}$

$= 100000\,e^{2\ln 1.5}$

$=$ ~~100 000 1.5~~

$= 225\,000$ ✓

QUESTION 9

a) $y = \log \frac{2x+1}{3x-7} = \log(2x+1) - \log 3x - 7$

$$y' = \frac{2}{2x+1} - \frac{3}{3x-7}$$

b) $V = 3 - \frac{2}{1+t}$

$x = 3t - 2\ln(1+t) + c$

$x = 1$ when $t = 0$

$\therefore 1 = 3(0) - 2(\ln 1) + C$

$1 = C$

(i) $\therefore x = 3t - 2\ln(1+t) + 1$

(ii) Since $\frac{2}{1+t}$ can never be 0, V will never be 3

(iii) $V = 3 - 2(1+t)^{-1}$

$a = 2(1+t)^{-2} = \frac{2}{(1+t)^2}$

$a(2) = \frac{2}{(1+2)^2} = \frac{2}{9}$ m/sec^2

c) (i) L.H.S. $(\operatorname{Cosec}^2 A - 1)\operatorname{Sin}^2 A$

$= \left(\frac{1}{\operatorname{Sin}^2 A} - 1\right) \cdot \operatorname{Sin}^2 A$

$= \frac{1 - \operatorname{Sin}^2 A}{\operatorname{Sin}^2 A} \cdot \operatorname{Sin}^2 A = 1 - \operatorname{Sin}^2 A = \operatorname{Cos}^2 A$

(ii) $\operatorname{Cos}^2 A = \frac{3}{4}$

$A = \operatorname{Cos}^{-1} \pm \frac{\sqrt{3}}{2}$

$A = \frac{\pi}{6}, \frac{-\pi}{6}$

QUESTION 10

a) $V = \pi r^2 h$ $\qquad$ $S.A = \pi r^2 + 2\pi r h$

$16 = \pi x^2 y$ $\qquad$ $= \pi x^2 + 2\pi x \cdot \frac{16}{\pi x^2}$

$\frac{16}{\pi x^2} = y$ ✓ $\qquad$ $= \pi x^2 + \frac{32}{x}$ ✓

b) $s = \left(\frac{3}{4}x\right)^2 + x^2$ $\qquad$ $S.A = \pi r s$

$= \frac{9}{16}x^2 + x^2$ $\qquad$ $= \pi \times x \cdot \frac{5x}{4}$

$= \frac{25x^2}{16}$ $\qquad$ $= \frac{5\pi x^2}{4}$

$s = \frac{5x}{4}$ ✓

c) $C = K \times \left(\pi x^2 + \frac{32}{x}\right) + 1.5K\left(\frac{5\pi x^2}{4}\right)$ or $\frac{3}{2}K\left(\frac{5\pi x^2}{4}\right.$

$= K\pi x^2 + \frac{32K}{x} + \frac{15K\pi x^2}{8}$

$= \frac{23K\pi x^2}{8} + \frac{32K}{x}$

$= K\left(\frac{23\pi x^2}{8} + \frac{32}{x}\right)$

d) $\frac{dc}{dx} = \frac{46K\pi x}{8} - \frac{32K}{x^2}$ $\qquad$ At min $\frac{dc}{dx} = 0$

$0 = \frac{46K\pi x^3 - 256K}{8x^2}$ $\qquad$ $x = \sqrt[3]{\frac{256}{46\pi}}$

$46\pi x^3 = 256$ $\qquad$ $= 1.210$